A Finite-Element Method for Calculating Aerodynamic Coefficients of a Subsonic Airplane

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A finite-element method, with sources and doublets representing the body and vortex-lattice representing lifting surfaces, has been developed for calculating both the longitudinal and the lateral aerodynamic coefficients of a subsonic airplane. This method has been verified with the wind-tunnel test data.

Nomenclature

μ = twist angle
ϵ = dihedral
Λ = sweep angle
α = angle of attack
β = sideslip angle
M = Mach number
ϕ' = pertubation potential of source
ϕ'' = pertubation potential of doublet
Γ = vortex strength
Γ' = source strength
Γ'' = doublet strength
$\mathbf{x}(x_1,x_2,x_3)$ = position vector
$\mathbf{x}_c$ = position vector of control point
$\mathbf{x}_L$ = position vector of load point
$\mathbf{x}_{C.G.}$ = position vector of airplane c.g.
ξ = position vector of source or doublet
$\mathbf{R}$ = $\mathbf{x}_c - \mathbf{x}$
$R_B{}^2$ = $(x_{c1} - x_1)^2 + (1 - M^2)[x_{c2} - x_2)^2 + (x_{c3} - x_3)^2]$
$\mathbf{d}$ = $(x_1 - \xi_1)i + (1 - M^2)[(x_2 - \xi_2)\mathbf{j} + (x_3 - \xi_3)\mathbf{k}$
d^2 = $(x_1 - \xi_1)^2 + (1 - M^2)[(x_2 - \xi_2)^2 + (x_3 - \xi_3)^2]$
(D_x,D_y,D_z) = $\mathbf{x}_L - \mathbf{x}_{CG}$ or $\xi - \mathbf{x}_{CG}$
$\mathbf{m}$ = unit vector of doublet
θ_0 = $\cot^{-1}(\sin\alpha \cot\beta)$ (see Fig. 2)
(x,r,θ) = surface coordinates of body
$\mathbf{V}_\infty$ = freestream velocity
$\mathbf{V}$ = induced velocity due to vortex
$\mathbf{V}'$ = induced velocity due to source
$\mathbf{V}''$ = induced velocity due to vortex
δ_{ij} = Kronecker delta
S = wing area
c = mean aerodynamic chord
b = wing span
l = half panel width

Introduction

THE finite-element method for evaluating aerodynamic forces of an airplane has made rapid progress since the introduction of the high speed computer. It has been reviewed by Landahl and Stark,[1] Ashley,[2] and Bradley and Miller.[3] In the steady aerodynamic forces area, Woodward,[4] Belotserkovskii,[5] Hedman[6] and Kalman et al.,[7] have made their prominent contributions. However, most of their works are only concerning the longitudinal forces and moment. A general consideration about all six aerodynamic coefficients of an airplane has not been found in the literature. Therefore, a finite-element method was developed for this purpose. By using this method, the designer can investigate various tentative configurations to find the one which has the best performance and flying qualities before a wind-tunnel test is started.

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The Method

In this method, the fuselage and nacelles are represented by sources and doublets, and the lifting surfaces are represented by vortex-lattices.

The panels on lifting surfaces are cut, as in the usual arrangement of the vortex-lattice method, with side edges parallel to the fuselage reference line, and the leading and the trailing edges accommodating the taper and the sweep of the surface. The control point is selected on the three quarter chord of the midspan of each panel. The unit normal of a panel is then (Fig. 1)

$$\boldsymbol{n}_p = \sin\mu \boldsymbol{i} - \cos\mu \sin\epsilon \boldsymbol{j} + \cos\mu \cos\epsilon \boldsymbol{k}$$

A bound vortex is attached to each panel at quarter chord position with a load point at midspan. The differential length vetor of bound vortex is

$$\begin{aligned} \boldsymbol{ds} &= [\tan\Lambda \cos\mu \boldsymbol{i} + (\cos\epsilon + \tan\Lambda \sin\mu \sin\epsilon)\boldsymbol{j} \\ &\quad + (\sin\epsilon - \tan\Lambda \sin\mu \cos\epsilon)\boldsymbol{k}]dh \\ &= adh \end{aligned} \tag{1}$$

The trailing vortices are set parallel to the wind. The differential length vector of a trailing vortex is then

$$\begin{aligned} \boldsymbol{ds} &= \pm (\boldsymbol{i} - \frac{\tan\beta}{\cos\alpha}\boldsymbol{j} + \tan\alpha \boldsymbol{k})dx_1 \\ &= \pm \boldsymbol{b} dx_1 \end{aligned} \tag{2}$$

The induced velocity calculated by using the compressible Biot-Savart Law is

$$\boldsymbol{dV} = -\frac{(1-M^2)\Gamma}{4\pi}\frac{\boldsymbol{R}\times\boldsymbol{ds}}{R_B{}^3} \tag{3}$$

The total induced velocity at panel i, generated by the vortex-lattice j, can be reduced (Appendix A) to the form as

$$V_i = u_{ij}\Gamma_j \tag{4}$$

The fuselage and nacelles are approximated by circular cones with bases normal to the fuselage reference line. A source as well as a doublet is attached on the center of each cone. The doublet is in the plane of cross section with angle $\Theta_0 = \cot^{-1}(\sin\alpha \cot\beta)$ to z-axis (Fig. 2). The unit vector in the direction of doublet is then $\mathbf{m} = \sin\Theta_0\mathbf{j} - \cos\Theta_0\mathbf{k}$. Two control points are chosen on the top and bottom surfaces of the body in the central plane with the normal $\mathbf{n}_u = -(dr/dx)\,\mathbf{i} - \mathbf{k}$ and $\mathbf{n}_l = -(dr/dx)\,\mathbf{i} + \mathbf{k}$ respectively. The pertubation potentials at the control point i, due to the source and doublet at point j, are

$$\phi_i' = -\Gamma_j'/d_{ij} \tag{5}$$

$$\phi_i'' = -[(m_i \cdot d_{ij})/d_{ij}^3]\Gamma_j'' \quad (6)$$

The induced velocity at control point i, generated by singularities at j, can be written (Appendix B) as follows:

$$V_i' = u_{ij}'\Gamma_j' \quad (7)$$

$$V_i'' = u_{ij}''\Gamma_j'' \quad (8)$$

The strength of the sources and doublets are first evaluated by satisfying the flow condition on the control points on fuselage and nacelles, i.e.

$$(V_i' + V_i'' + V_\infty) \cdot n_{ui} = 0$$

$$(V_i' + V_i'' + V_\infty) \cdot n_{li} = 0$$

or in matrix form

$$\left[\begin{array}{c|c} \delta_{im}u_{ijk}'n_{umk} & \delta_{im}u_{ijk}''n_{umk} \\ \hline \delta_{im}u_{ijk}'n_{lmk} & \delta_{im}u_{ijk}''n_{lmk} \end{array}\right]\left\{\begin{array}{c} \Gamma_j' \\ \hline \Gamma_j'' \end{array}\right\} = -\left\{\begin{array}{c} V_{\infty k}n_{uki} \\ \hline V_{\infty k}n_{lki} \end{array}\right\} \quad (9)$$

where k = 1, 2, 3. These equations are solved for Γ_j' and Γ_j''. The strength of horseshoe vortices on lifting surface are then calculated by satisfying the flow condition on the control points including the influence of the fuselage and nacelles.

$$(V_i + V_i' + V_i'' + V_\infty) \cdot n_{pi} = 0$$

or in matrix form

$$[\delta_{im}u_{ijk}n_{pmk}]\{\Gamma_j\} =$$

$$-\{V_{\infty k}n_{pki}\} - [\delta_{im}u_{ink}'n_{pmk} \;\; \delta_{im}u_{ink}''n_{pmk}]\left\{\begin{array}{c}\Gamma_n' \\ \hline \Gamma_n''\end{array}\right\} \quad (10)$$

which can be solved for Γ_j.

The differential force generated by a vortex of strength Γ and length **ds** in a flow of velocity **V** is $\mathbf{dF} = \rho\mathbf{V} \times \Gamma\mathbf{ds}$. Therefore, we can calculate the force generated on each lifting panel from the known induced velocity due to Γ_j, Γ_j' and Γ_j'', as

$$F_i = \rho(p_{xi}i + p_{yi}j + p_{zi}k) \quad (11)$$

where

$$p_{xi} = 2\Gamma_i l_i[(-V_\infty \sin\beta + v_{2i})(a_{3i} - a_{1i}b_3)$$
$$-(V_\infty \cos\beta \sin\alpha + v_{3i})(a_{2i} - a_{1i}b_2)]$$

$$p_{yi} = -2\Gamma_i l_i(V_\infty \cos\beta \cos\alpha + v_{1i})(a_{3i} - a_{1i}b_3)$$

$$p_{zi} = 2\Gamma_i l_i(V_\infty \cos\beta \cos\alpha + v_{1i})(a_{2i} - a_{1i}b_2)$$

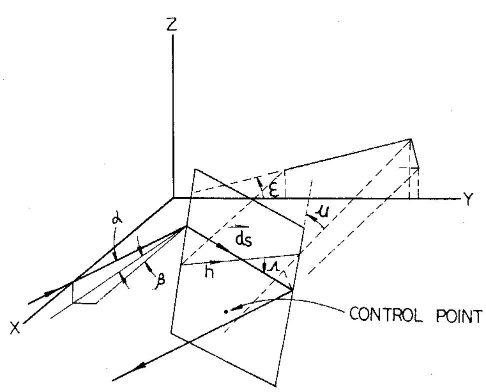

Fig. 1 Lift surface with horseshoe vortex and control point.

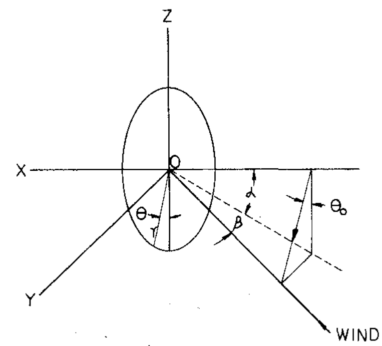

Fig. 2 Coordinates of body surface and doublet.

v_{1i} v_{2i} v_{3i} are induced velocity components due to vortices, sources and doublets.

The forces exerted on the fuselage and nacelles are calculated, by integrating the pressure coefficient due to sources and doublets around each cone, with the assumption that the vortices on lifting panels give negligible contribution to the total force on a cone. For an axisymmetric body with angle of attack α and yaw angle β the pressure coefficient at surface (r,Θ,x) is

$$C_p = -\frac{2u_x \cos\alpha \cos\beta}{V_\infty} + \frac{2\sin\alpha}{V_\infty \cos\Theta_0}[u_r \cos(\Theta - \Theta_0)$$
$$-u_\Theta \sin(\Theta - \Theta_0)] - \frac{u_r^2 + u_\Theta^2}{V_\infty^2} \quad (12)$$

Then the force exerted on i section of body equals

$$F_i' = \frac{\rho}{2}V_\infty^2 \int C_p n_i dA = \rho(p_{xi}'i + p_{yi}'j + p_{zi}'k) \quad (13)$$

n_i = inward unit normal

$$= [(\frac{dr}{dx})_i i - \sin\Theta_i j + \cos\Theta_i k]/[(\frac{dr}{dx})_i^2 + 1]^{1/2}$$

dA = differential area over the surface

$$= [(\frac{dr}{dx})_i^2 + 1]^{1/2} dx_i \cdot r_i d\Theta_i$$

The integral $\int C_p \mathbf{n}_i dA$ can be evaluated in close form in case the singularities are distributed along the centerline of the fuselage. For the other cases, a numerical integration shall be undertaken.

The aerodynamic coefficients can then be calculated as follows

$$C_L = \frac{2}{\rho V_\infty^2 S}(\sum_i F_i + \sum_j F_j') \cdot (\cos\alpha k - \sin\alpha i) \quad (14a)$$

$$C_D = \frac{2}{\rho V_\infty^2 S}(\sum_i F_i + \sum_j F_j') \cdot (\sin\alpha k + \cos\alpha i) \quad (14b)$$

$$C_Y = \frac{2}{\rho V_\infty^2 S}(\sum_i F_i + \sum_j F_j') \cdot j \quad (14c)$$

$$C_M = \frac{2}{\rho V_\infty^2 Sc}[(\sum_i F_i D_{zi} + \sum_j F_j' D_{zj}) \cdot i - (\sum_i F_i D_{xi}$$
$$+ \sum_j F_j' D_{xj}) \cdot k] \quad (14d)$$

$$C_l = \frac{2}{\rho V_\infty^2 Sb}\{(\sum_i F_i D_{yi} + \sum_j F_j' D_{yj}) \cdot (\sin\alpha i - \cos\alpha k)$$
$$+ [(\sum_i F_i D_{zi} + \sum_j F_j' D_{zj})\cos\alpha - (\sum_i F_i D_{xi}$$
$$+ \sum_j F_j' D_{xj})\sin\alpha] \cdot j \quad (14e)$$

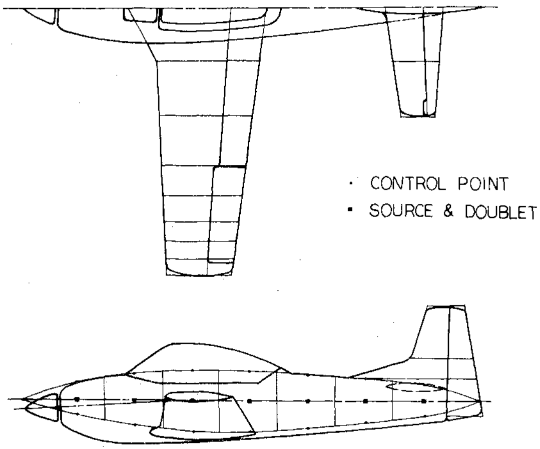

Fig. 3 Arrangement of elements.

$$C_N = -\frac{2}{\rho V_\infty^2 Sb}\{(\sum_i \boldsymbol{F}_i D_{yi} + \sum_j \boldsymbol{F}_j' D_{yj}) \cdot (\sin\alpha \boldsymbol{k} + \cos\alpha \boldsymbol{i}) + [(\sum_i \boldsymbol{F}_i D_{xi} + \sum_j \boldsymbol{F}_j' D_{xj})\cos\alpha + (\sum_i \boldsymbol{F}_i D_{zi} + \sum_j \boldsymbol{F}_j' D_{zj})\sin\alpha] \cdot \boldsymbol{j}\} \quad (14f)$$

This method has been used to predict aerodynamic coefficients of a turboprop trainer. The panels of lifting surfaces and the cones of body are arranged as shown in Fig. 3. The calculated data were then compared against the data of the wind-tunnel test which was conducted in the Beech Memorial Wind Tunnel of Wichita State University.[8] The results are generally good as shown in Figs. 4–8, although the maximum number of vortices employed in the program is only twenty-two and that of singularities along the fuselage is only fourteen, limited by the capacity of an IBM 1130 computer. It was suspected in the early stage of verification that the promising result was only a coincidence. Consequently, various combinations of panel arrangements were employed in calculation. The result showed convergence (Fig. 9), which agrees with what Deyoung[9] and James[10] have concluded through their theoretical investigation on vortex-lattice technique. The arrangement of singularities in body has also been tested numerically. The result did show improvement (Fig. 10), especially near the stagnation point of the body, by comparing the calculated pressure coefficients with Revell's 2nd order result.[11] It is conceivable that the outcome of the present method will be better if computed with more elements, though the induced drag prediction might not be improved without modifying the method by including leading edge suction.

As the computerized systems for preliminary design are under development in many branches of aircraft industry[12,13] and the existing methods for predicting aerodynamic coefficients are not very satisfactory,[14]

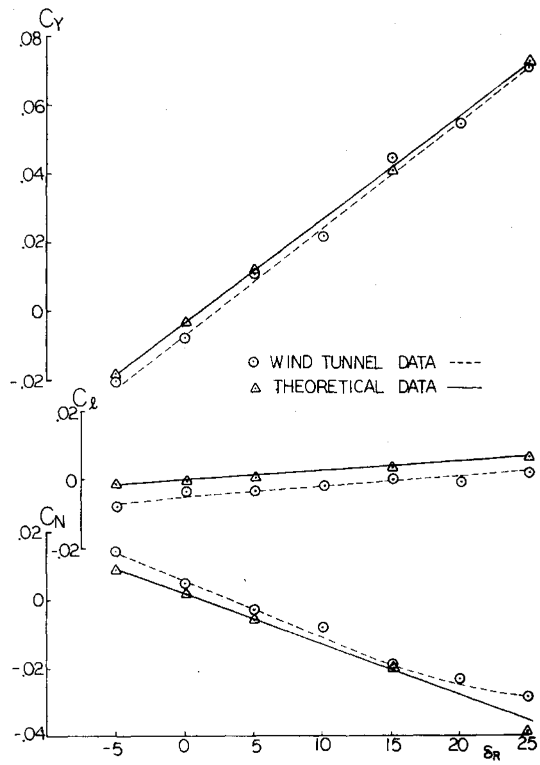

Fig. 5 Lateral coefficients with rudder deflected.

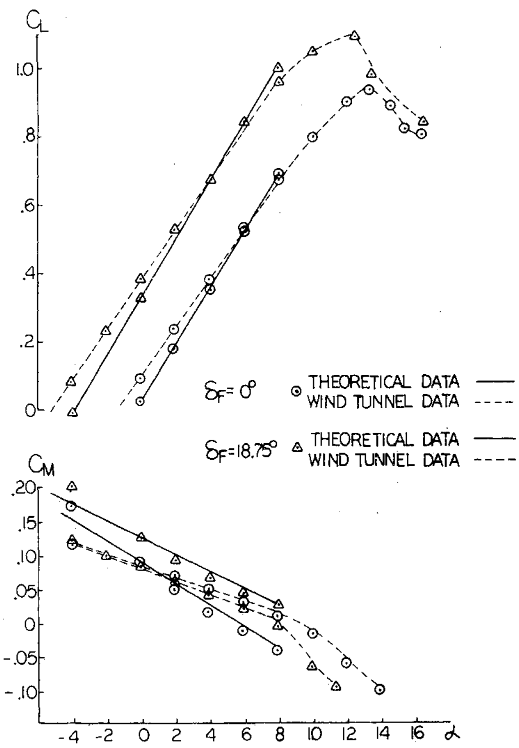

Fig. 4 C_L and C_M with and without flap deflected.

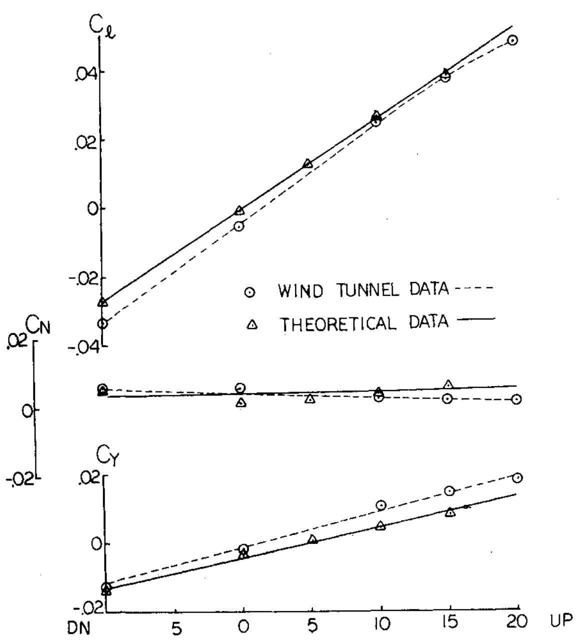

Fig. 6 Lateral coefficients with right aileron deflected.

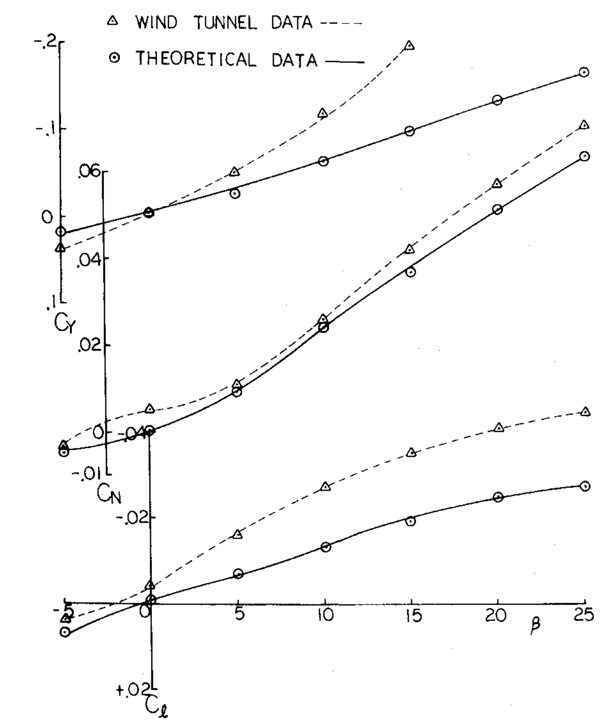

Fig. 7 Lateral coefficients at sideslip, $\alpha = 2°$.

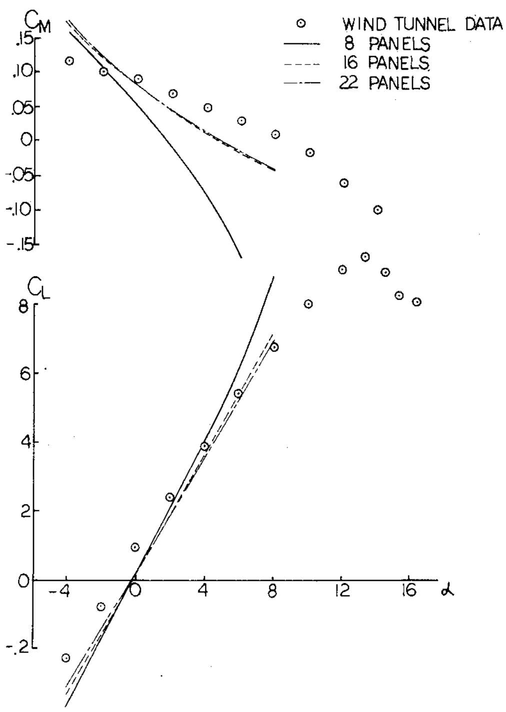

Fig. 9 Improvement by increasing number of lifting panels.

the present method, with further improvement in the number of elements used and in the drag computation, may perform as an appropriate aerodynamic module of a computerized preliminary design system.

Appendix A: Derivation of Eq. (4)

The distance vector from the differential vortex **ds** at **x** along a bound vortex to a control point $\mathbf{x}_c$

$$\boldsymbol{R} = x_c - x_L - ah = \boldsymbol{D} - ah \quad \text{(A1)}$$

Then

$$\boldsymbol{R} \times ds = (\boldsymbol{D} \times a)dh \quad \text{(A2)}$$

and

$$R_B^2 = c_1 + c_2 h + c_3 h^2 \quad \text{(A3)}$$

where

$$c_1 = D_1^2 + (1 - M^2)(D_2^2 + D_3^2)$$
$$c_2 = -2[a_1 D_1 + (1 - M^2)(a_2 D_2 + a_3 D_3)]$$
$$c_3 = a_1^2 + (1 - M^2)(a_2^2 + a_3^2)$$

The induced velocity at a control point due to a bound vortex is

$$\boldsymbol{V}_B = -\frac{(1 - M^2)}{4\pi}\left[\frac{c_2 + 2c_3 h}{(4c_1c_3 - c_2^2)(c_1 + c_2 h + c_3 h^2)^{1/2}}\right]_{-l}^{l} \cdot (\boldsymbol{D} \times a)\Gamma$$
$$= u_B \Gamma \quad \text{(A4)}$$

Similarly along the left and right trailing vortices

$$\boldsymbol{R} = (\boldsymbol{D} \pm la) - [x - (x_L \mp la)] \quad \text{(A5)}$$

$$\boldsymbol{R} \times ds = (\boldsymbol{D} \pm la) \times b\, dx_1 \quad \text{(A6)}$$

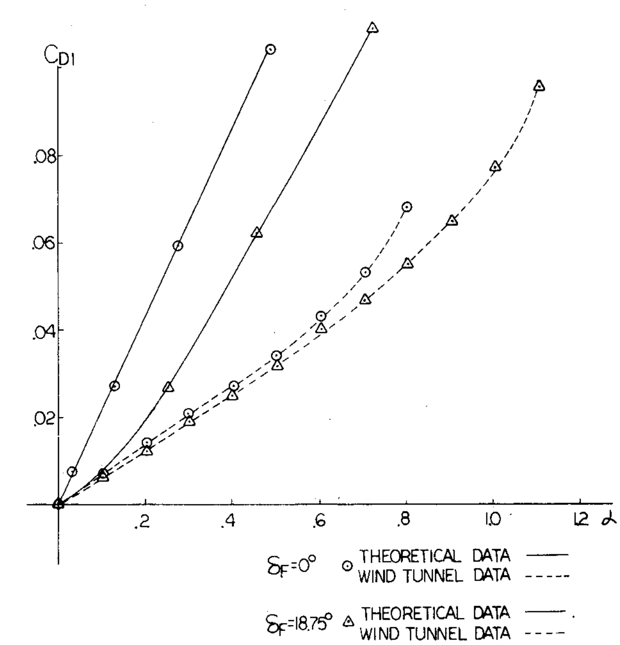

Fig. 8 Induced drag coefficients.

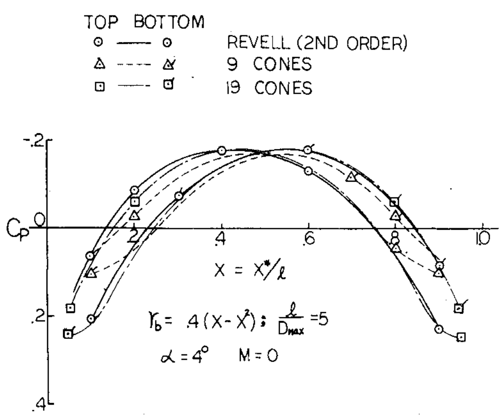

Fig. 10 Improvement by increasing number of cones.

From the relation

$$x_i - (x_{Li} \mp la_i) = b_i[x_1 - (x_{L1} \mp la_1)]; \quad i = 2,3 \quad \text{(A7)}$$

we can write

$$R_B^2 = e_1 + e_2x_1 + e_3x_1^2 \quad \text{(A8)}$$

where

$$e_1 = x_{c1}^2 + \{[D_2 \pm la_2 + b_2(x_{L1} \mp la_1)]^2 + [D_3 \pm la_3 + b_3(x_{L1} \mp la_1)]^2\}(1 - M^2)$$

$$e_2 = -2\{x_{c1} + [b_2(D_2 \pm la_2 + b_2x_{L1} \mp b_2la_1) + b_3(D_3 \pm la_3 + b_3x_{L1} \mp b_3la_1)] \cdot (1 - M^2)\}$$

$$e_3 = 1 + (1 - M^2)(b_2^2 + b_3^2)$$

The induced velocity due to trailing vortices are then

$$\boldsymbol{V}_T = \pm \frac{(1-M^2)}{4\pi}\left[\frac{e_2 + 2e_3x_1}{(4e_1e_3 - e_2^2)(e_1 + e_2x_1 + e_3x_1^2)^{1/2}}\right]_{x_1 \mp la_1}^{\infty} \cdot (\boldsymbol{D} \pm l\boldsymbol{a}) \times \boldsymbol{b}\,\Gamma$$
$$= u_L\Gamma \ \text{(left trailing vortex)}$$
$$= u_R\Gamma \ \text{(right trailing vortex)} \quad \text{(A9)}$$

The total induced velocity at i control point due to all j

$$\boldsymbol{V}_i = (u_{Bij} + u_{Lij} + u_{Rij})\Gamma_j = u_{ij}\Gamma_j$$

Appendix B: Derivation of Eqs. (7) and (8)

For the sources of strength Γ_j' located at ξ_j, the induced velocity at $\mathbf{x}_i$ in the compressible flow is

$$\boldsymbol{V}_i' = \frac{\partial}{\partial x_i}(\phi_i') = \frac{d_{ij}}{d_{ij}^3}\Gamma_j' \quad \text{(A10)}$$

Let

$$u_{ij}' = \frac{d_{ij}}{d_{ij}^3}$$

For the doublets of strength Γ_j'' in the direction of $\mathbf{m}_j$, located at ξ_j, the induced velocity at $\mathbf{x}_i$ in the compressible flow is

$$\boldsymbol{V}_i'' = \frac{\partial}{\partial x_i}(\phi_i'')$$
$$= \{-(1-M^2)(\sin\Theta_o j - \cos\Theta_o k)\frac{1}{d_{ij}^3} + 3(1-M^2)[(x_{2i} - \xi_{2j})\sin\Theta_o - (x_{3i} - \xi_{3j})\cos\Theta_o] \cdot [(x_{1i} - \xi_{1j})i + (1-M^2)(x_{2i} - \xi_{2j})j + (1-M^2)(x_{3i} - \xi_{3j})k]/d_{ij}^5\}\Gamma_j'' \quad \text{(A11)}$$

Let the terms in the braces be $\mathbf{u}_{ij}''$, then

$$\boldsymbol{V}_i'' = u_{ij}''\Gamma_j''$$

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